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$g/u(1)^d$ parafermions from constrained WZNW theories

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Abstract. Operator quantization of the WZNW theory for the affine Lie algebra $\hat{g}$ with the constrained $\hat{u}(1)^d$ subalgebra is performed using a modification of the generalized canonical quantization method. The constrained theory is equivalent to the $g/u(1)^d$ coset conformal field theory. Realizations of the $g/u(1)^d$ parafermions for arbitrary d are found.

1. Introduction

The $g/u(1)^d$, $1 \leqslant d \leqslant \text{rank}\, g$, parafermions are generators of an extended symmetry algebra of the conformal field theory based on the $g/u(1)^d$ coset construction [1, 2]

$$K = L^g - L^{u(1)^d} \tag{1}$$

where L^g is the energy–momentum tensor, which is associated with the affine Lie algebra $\hat{g}$ via the Sugawara formula. In the case of $su(2)/u(1)$ the parafermions were constructed by Fateev and Zamolodchikov [3]. The parafermion algebra of the $g/u(1)^r$, $r = \text{rank}\, g$, coset for an arbitrary simple Lie algebra g was found in [4]. The vertex operator construction for primary fields of $g/u(1)^r$ was obtained in [5].

In a recent article [6] we displayed the approach to the study of the $g/u(1)^d$ conformal field theory which is based on the connection between this theory and the Wess–Zumino–Novikov–Witten (WZNW) theory for $\hat{g}$ with the constrained $\hat{u}(1)^d$ currents. Using an operator version of Dirac's procedure [7] it was shown that the initial energy–momentum tensor L^g is replaced in the constrained theory by K. It turned out that the constrained $su(2)$ WZNW theory is equivalent to the $su(2)/u(1)$ theory.

In the present paper we generalize this result to the $g/u(1)^d$, $1 \leqslant d \leqslant r$, theory. We perform operator quantization of the constrained WZNW theory using the approach which is based on a modification of the generalized canonical quantization method (see, e.g., [8] and references therein) and show that the resulting symmetry algebra is an extended symmetry algebra of the $g/u(1)^d$ coset conformal field theory. We present two explicit realizations for the parafermions: a realization where they are represented as elements of a quotient algebra and a vertex operator construction similar to that of [4]. Using the second realization we find the equations which define the $g/u(1)^d$ parafermions through their relationship with the generators of $\hat{g}$. The theory has a symmetry group which is generated by the $u(1)^r$ currents.

The parafermionic algebra is associated with a classical quadratically nonlinear algebra which can be obtained using both the generalized canonical quantization method and the Dirac-bracket formalism.

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The plan of the paper as follows. In section 2 we describe the quantization method. Putting auxiliary operators into correspondence with first and second-class constraints we convert the initial constraints into effective abelian constraints. The constrained algebra is obtained by replacing the initial operators by the operators which commute with the effective abelian constraints. In section 3 we review the WZNW theory and introduce the constraints. In section 4 we perform quantization of the constrained WZNW theory and find the operators which replace the affine currents. These operators are expressed in terms of the original affine currents and free bosons. We show that the $g/u(1)^d$ parafermion algebra can be formulated as a quotient algebra in a natural way and find the parafermion currents. In section 5 we find the vertex operator construction for the parafermion algebra. In section 6 we consider the constrained classical current algebra, find the corresponding Dirac brackets and show that the same brackets can be obtained using the generalized canonical quantization method.

2. The quantization method

Let T^A and T_μ be first- and second-class constraints. For our purpose it is sufficient to consider constraints of zero Grassmann parity $\epsilon(T^A) = \epsilon(T_\mu) = 0$ and assume that they obey the commutator relations

$$[T^A, T^B] = [T^A, T_\mu] = 0 \qquad [T_\mu, T_\nu] = r_{\mu\nu} \tag{2}$$

where $r_{\mu\nu}$ is an invertible c-number matrix.

According to the prescription of the generalized canonical quantization method [8] we introduce the auxiliary operators b_μ, which obey the commutator relation

$$[b_\mu, b_\nu] = r_{\mu\nu} \tag{3}$$

and convert the second-class constraints T_μ into the effective abelian constraints

$$\tilde{T}_\mu(b) = T_\mu + \mathrm{i}b_\mu. \tag{4}$$

It is easy to see that

$$[\tilde{T}_\mu, \tilde{T}_\nu] = 0 \qquad \tilde{T}_\mu(0) = T_\mu \tag{5}$$

and

$$[\tilde{T}_\mu, T^A] = 0. \tag{6}$$

Let us put into correspondence with each constraint T^A a pair of the canonical operators (p^A, q^A), which, in contrast with [8], are supposed to be of the same Grassmann parity as the constraints. In the extended phase space the operator T^A can be replaced by the operator $\tilde{T}^A(\chi)$, $\chi = (b_\mu, p^A, q^A)$, which, as well as T^A, satisfies the equations

$$[\tilde{T}_\mu, \tilde{T}^A] = 0 \qquad [\tilde{T}^A, \tilde{T}^B] = 0 \tag{7}$$

and the boundary conditions

$$\tilde{T}^A(0) = T^A. \tag{8}$$

Let F be an original operator. We shall demand [8] that F is replaced in the constrained theory by the operator $\mathcal{F}(\chi)$ which commutes with the effective abelian constraints $(\tilde{T}^A, \tilde{T}_\mu)$. The generating equations of the constrained algebra then assume the form

$$[\tilde{T}_\mu, \mathcal{F}] = 0 \qquad [\tilde{T}^A, \mathcal{F}] = 0 \qquad \mathcal{F}(0) = F. \tag{9}$$

The solutions of equations (7) and (9) are not completely fixed by the boundary conditions. The arbitrariness is to be removed by a gauge condition.

3. The WZNW model and constraints

The WZNW theory [9, 10] is invariant with respect to two commuting affine Lie algebras, which are generated by the left and right currents $(H^s(z), E^\alpha(z))$, $(\bar{H}^s(\bar{z}), \bar{E}^\alpha(\bar{z}))$, where $s = 1, \ldots, r$ and α are roots of g. Since the algebras generated by the left and right currents are identical we shall only consider the first. The theory is also invariant with respect to the conformal algebra. The Virasoro generator $L^g(z)$ is expressed quadratically in terms of the currents.

The fields $H^s(z)$, $E^\alpha(z)$ and $L^g(z)$ satisfy the operator product expansions

$$
\begin{aligned}
&H^s(z)H^t(w) = \frac{k\delta^{st}}{(z-w)^2} \qquad H^s(z)E^\alpha(w) = \frac{\alpha^s E^\alpha(w)}{z-w} \\
&E^\alpha(z)E^\beta(w) = \begin{cases} \dfrac{f(\alpha,\beta)E^{\alpha+\beta}(w)}{z-w} & \text{if} \quad \alpha+\beta \text{ is a root} \\ \dfrac{k}{(z-w)^2} + \dfrac{\alpha H(w)}{z-w} & \text{if} \quad \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \\
&L^g(z)H^s(w) = \frac{H^s(w)}{(z-w)^2} + \frac{\partial_w H^s(w)}{z-w} \\
&L^g(z)E^\alpha(w) = \frac{E^\alpha(w)}{(z-w)^2} + \frac{\partial_w E^\alpha(w)}{z-w} \\
&L^g(z)L^g(w) = \frac{c_g}{2(z-w)^4} + \frac{2L^g(w)}{(z-w)^2} + \frac{\partial_w L^g(w)}{z-w} \\
&c_g = \frac{k \dim g}{k+h^\vee}
\end{aligned}
\tag{10}
$$

where k is the level of the representation, $f(\alpha,\beta)$ are the structure constants and $h^\vee$ is the dual Coxeter number of the algebra g.

Let g be simply laced. In this case the currents $H^s(z)$, $E^\alpha(z)$ and energy–momentum tensor $L^g(z)$ can be expressed in terms of the bosonic fields

$$
\varphi^{sj}(z) = x^{sj} - \mathrm{i}a_0^{sj}\log z + \mathrm{i}\sum_{n\neq 0}\frac{a_n^{sj}}{n}z^{-n}
\tag{11}
$$

where $j = 1, \ldots, k$ and

$$
[x^{sj}, a_0^{tl}] = \mathrm{i}\delta^{st}\delta^{jl} \qquad [a_m^{sj}, a_n^{tl}] = m\delta^{st}\delta^{jl}\delta_{m+n,0}.
\tag{12}
$$

The bosonic constructions for $H^s(z)$, $E^\alpha(z)$, $L^g(z)$ and $L^{u(1)^d}(z)$ read [4, 11, 12]

$$
\begin{aligned}
&E^\alpha(z) = \sum_{j=1}^{k} :\mathrm{e}^{\mathrm{i}\alpha\cdot\varphi^j(z)}: c_\alpha^j \qquad H^s(z) = \sum_{j=1}^{k}\mathrm{i}\partial_z\varphi^{sj}(z) \\
&L^g(z) = \frac{1}{2(k+h^\vee)}\Bigg((1+h^\vee)\sum_{s=1}^{r}\sum_{j=1}^{k} :(\mathrm{i}\partial_z\varphi^{sj})^2: +2\sum_{s=1}^{r}\sum_{i<j}^{k} :(\mathrm{i}\partial_z\varphi^{sj})(\mathrm{i}\partial_z\varphi^{sj}): \\
&\qquad +2\sum_{\alpha}\sum_{i<j}^{k} :\exp(\mathrm{i}\alpha\cdot(\varphi^i-\varphi^j)): c_\alpha^i c_{-\alpha}^j\Bigg) \\
&L^{u(1)^d}(z) = \frac{1}{2k}\sum_{A=1}^{d} :\left(\sum_{j=1}^{k}\mathrm{i}\partial_z\varphi^{Aj}(z)\right)^2:
\end{aligned}
\tag{13}
$$

where : : denotes normal ordering with respect to the modes of the bosons, $\alpha^2 = 2$ and c_α^j is a cocycle operator.

We shall consider the WZNW theory subject to the constraints

$$H^A(z) \approx 0 \tag{14}$$

where $A = 1, \ldots, d$. It is convenient to decompose the constraints into modes

$$H^A(z) = H_0^A z^{-1} + \sum_{n\neq 0} H_n^A z^{-n-1} \qquad H_0^A = \sum_{j=1}^{k} a_0^{Aj} \qquad H_n^A = \sum_{j=1}^{k} a_n^{Aj} \tag{15}$$

and consider an equivalent set of constraints:

$$H_0^A \approx 0 \qquad H_n^A \approx 0. \tag{16}$$

The operators H_0^A, H_n^A obey the algebra

$$[H_0^A, H_0^B] = [H_0^A, H_n^B] = 0 \qquad [H_m^A, H_n^B] = km\delta^{AB}\delta_{m+n,0}. \tag{17}$$

It follows from these commutator relations that the constraints H_0^A are first class and H_n^A, $n \neq 0$, are second class.

4. Quantization of the model

4.1. Generating equations of the constrained symmetry algebra

To quantize the system we put into correspondence with the constraints the operators $\chi = (b_n^A, n \neq 0, p^A, q^A)$ which satisfy the commutator relations

$$[b_m^A, b_n^B] = km\delta^{AB}\delta_{m+n,0} \qquad [q^A, p^B] = \mathrm{i}k\delta^{AB}. \tag{18}$$

The effective abelian constraints $\tilde{H}_n^A(b)$ are given by

$$\tilde{H}_n^A = H_n^A - \mathrm{i}b_n^A \tag{19}$$

and we replace the first class constraints H_0^A by the following:

$$\tilde{H}_0^A = H_0^A - \mathrm{i}p^A. \tag{20}$$

According to equations (9) the other generators of the constrained symmetry algebra $\mathcal{H}^P(z) \equiv \mathcal{H}^P(\chi, z)$, where $P = d+1, \ldots, r$, $\mathcal{E}^\alpha(z) \equiv \mathcal{E}^\alpha(\chi, z)$ and $\mathcal{L}^g(z) \equiv \mathcal{L}^g(\chi, z)$ satisfy the equations

$$[\tilde{H}_N^A, \mathcal{H}^P] = 0 \qquad [\tilde{H}_N^A, \mathcal{E}^\alpha] = 0 \qquad [\tilde{H}_N^A, \mathcal{L}^g] = 0 \qquad N \in Z \tag{21}$$

with the boundary conditions

$$\mathcal{H}^P(0, z) = H^P(z) \qquad \mathcal{E}^\alpha(0, z) = E^\alpha(z) \qquad \mathcal{L}^g(0, z) = L^g(z). \tag{22}$$

A solution of these equations is given by

$$\begin{aligned} &\mathcal{H}^P(z) = H^P(z) \qquad \mathcal{E}^\alpha(z) = E^\alpha(z) : \mathrm{e}^{\frac{1}{k}\tilde{\alpha}\cdot\phi(z)} : \\ &\mathcal{L}^g(z) = K(z) + \frac{1}{2k}\sum_{A=1}^{d}(\tilde{H}^A(z))^2 \end{aligned} \tag{23}$$

where $\tilde{\alpha} = (\alpha^A)$,

$$\tilde{H}^A(z) = H^A(z) + \partial_z\phi^A(z) \tag{24}$$

and $\phi = (\phi^A)$ are the bosonic fields:

$$\phi^A(z) = q^A - \mathrm{i}p^A \log z + \mathrm{i}\sum_{n\neq 0} \frac{b_n^A}{n} z^{-n}. \tag{25}$$

This solution is not the general one. It can be fixed by a gauge condition.

4.2. Parafermion currents

The operator product expansions of K with $\mathcal{H}^P$ and $\mathcal{E}^\alpha$ are given by

$$\begin{aligned} K(z)\mathcal{H}^P(w) &= \frac{\mathcal{H}^P(w)}{(z-w)^2} + \frac{\partial_w \mathcal{H}^P(w)}{z-w} \\ K(z)\mathcal{E}^\alpha(w) &= \frac{\Delta_\alpha \mathcal{E}^\alpha(w)}{(z-w)^2} + \frac{1}{z-w}\left(\partial_w \mathcal{E}^\alpha(w) - \frac{1}{k}\tilde{\alpha}\tilde{H}(w)\mathcal{E}^\alpha(w)\right) \end{aligned} \tag{26}$$

where

$$\Delta_\alpha = 1 - \frac{\tilde{\alpha}^2}{2k}. \tag{27}$$

Let Ω be the algebra generated by $\mathcal{H}^P(z)$, $\mathcal{E}^\alpha(z)$ and $K(z)$. Since these operators commute with $\tilde{H}_N^A$ so therefore must all the fields of Ω. Let us define the following set of fields:

$$\Upsilon = (U \in \Omega | U|_{\tilde{H}^A=0} = 0). \tag{28}$$

An arbitrary field $U(z) \in \Upsilon$ can be written in the form

$$U(z) = \sum_A \sum_N \tilde{H}_N^A U_N^A(z) \tag{29}$$

where $U_N^A(z)$ commute with $\tilde{H}_N^A$. For an arbitrary $X(z) \in \Omega$ we have

$$U(z)X(w) \in \Upsilon \qquad X(z)U(w) \in \Upsilon. \tag{30}$$

Hence Υ is an ideal of Ω and the quotient Ω/Υ is an algebra. We shall denote by $\{X(z)\}$ the coset represented by the field $X(z)$.

It is easy to see that the field $\{K(z)\}$ satisfies the $g/u(1)^d$ Virasoro algebra

$$\{K(z)\}\{K(w)\} = \frac{c_{g/u(1)^d}}{2(z-w)^4} + \frac{2\{K(w)\}}{(z-w)^2} + \frac{\partial_w\{K(w)\}}{z-w} \tag{31}$$

where $c_{g/u(1)^d} = c_g - d$. It follows from (23) that $\{K(z)\}$ can be represented by $\mathcal{L}^g(z)$ as well. Equations (26) express the fact that $\{\mathcal{H}^P(z)\}$ and $\{\mathcal{E}^\alpha(z)\}$ are primary fields of the $g/u(1)^d$ Virasoro algebra:

$$\begin{aligned} \{K(z)\}\{\mathcal{H}^P(w)\} &= \frac{\{\mathcal{H}^P(w)\}}{(z-w)^2} + \frac{\partial_w\{\mathcal{H}^P(w)\}}{z-w} \\ \{K(z)\}\{\mathcal{E}^\alpha(w)\} &= \frac{\Delta_\alpha\{\mathcal{E}^\alpha(w)\}}{(z-w)^2} + \frac{\partial_w\{\mathcal{E}^\alpha(w)\}}{z-w}. \end{aligned} \tag{32}$$

The theory has the discrete symmetry group which is generated by H_0^s:

$$[H_0^s, \{K\}] = 0 \qquad [H_0^s, \{\mathcal{E}^\alpha\}] = \alpha^s\{\mathcal{E}^\alpha\}. \tag{33}$$

Let us consider the case of $su(2)/u(1)$. The $\widehat{su}(2)$ generators $E^+(z)$, $E^-(z)$ and $H(z)$ are given by

$$E^+(z) = \sum_{j=1}^k :\mathrm{e}^{\mathrm{i}\sqrt{2}\varphi^j(z)}: \qquad E^-(z) = \sum_{j=1}^k :\mathrm{e}^{-\mathrm{i}\sqrt{2}\varphi^j(z)}: \qquad H(z) = \sum_{j=1}^k \mathrm{i}\partial_z\varphi^j(z) \tag{34}$$

where $\varphi^j \equiv \varphi^{1j}$. In the theory with the constrained current $H(z) \approx 0$ these operators are replaced by

$$\begin{aligned} &\mathcal{E}^+(z) = E^+(z) :\mathrm{e}^{\frac{\sqrt{2}}{k}\phi(z)}: \qquad \mathcal{E}^-(z) = E^-(z) :\mathrm{e}^{-\frac{\sqrt{2}}{k}\phi(z)}: \\ &\tilde{H}(z) = H(z) + \partial_z\phi(z). \end{aligned} \tag{35}$$

We find that

$$\mathcal{E}^+(z)\mathcal{E}^-(w) = k(z-w)^{-2+\frac{2}{k}}\left(I + \frac{\sqrt{2}}{k}\tilde{H}(w)(z-w) + \left(\frac{k+2}{k}K(w) + \frac{\sqrt{2}}{2k}\partial_w\tilde{H}(w) + \frac{1}{k^2}\tilde{H}^2(w)\right)(z-w)^2 + \mathrm{O}((z-w)^3)\right). \qquad (36)$$

From this it follows that the currents $\{\mathcal{E}^+(z)\}$ and $\{\mathcal{E}^-(z)\}$ satisfy the parafermion algebra of [3]

$$\{\mathcal{E}^+(z)\}\{\mathcal{E}^-(w)\} = k(z-w)^{-2+\frac{2}{k}}\left(\{I\} + \frac{k+2}{k}\{K(w)\}(z-w)^2 + \mathrm{O}((z-w)^3)\right). \qquad (37)$$

5. Vertex operator construction for the parafermion algebra

In this section we show that the $g/u(1)^d$ theory for g simply laced can be formulated in terms of the original operators x^{sj}, a_0^{sj} and a_n^{sj}.

An arbitrary field $X(z) \in \Omega$ can be written in the form

$$X(z) = X_0(z) + \sum_A \sum_N \tilde{H}_N^A X_N^A(z) \qquad (38)$$

where $X_0 = X|_{\tilde{H}^A=0}$ and $X_N^A(z)$ commute with $\tilde{H}_N^A$. Let us define the function $f(X_0) = \{X\} = \{X_0\}$. By construction this is a homomorphism. It is easy to see that $f^{-1}(0) = 0$. Hence we obtain an isomorphism of Ω/Υ and the algebra $f^{-1}(\Omega/\Upsilon)$.

It is clear that $K_0 = K$ and $H_0^P = H^P$. We find

$$\mathcal{E}_0^\alpha(z) = \tilde{\mathcal{E}}^\alpha(z)\mathrm{e}^{\frac{1}{k}\tilde{\alpha}q}. \qquad (39)$$

Here $q = (q^A)$,

$$\tilde{\mathcal{E}}^\alpha(z) = \sum_{j=1}^k :\mathrm{e}^{\mathrm{i}[\tilde{\alpha}\omega^j(z)+\tilde{\alpha}^\perp\varphi^j(z)]}: c_\alpha^j \qquad \tilde{\alpha}^\perp = \alpha - \tilde{\alpha} \qquad (40)$$

$$\omega^{Aj}(z) = x^{Aj} - \mathrm{i}\sum_{l=1}^k \eta^{jl}\left(a_0^{Al}\log z - \sum_{n\neq 0}\frac{a_n^{Al}}{n}z^{-n}\right) \qquad (41)$$

and

$$\eta^{ij} = \begin{cases} \dfrac{(k-1)}{k} & \text{if} \quad i = j \\ -\dfrac{1}{k} & \text{if} \quad i \neq j. \end{cases} \qquad (42)$$

We have checked that $\mathcal{E}_0^\alpha(z)$ satisfies the equation

$$K(z)\mathcal{E}_0^\alpha(w) = \frac{\Delta_\alpha\mathcal{E}_0^\alpha(w)}{(z-w)^2} + \frac{\partial_w\mathcal{E}_0^\alpha(w)}{z-w}. \qquad (43)$$

Since q commutes with all the original operators we can set $q = 0$ in (39). Therefore the generators of Ω/Υ can be represented by the fields K, H^P and $\tilde{\mathcal{E}}^\alpha$, which are expressed in terms of x^{sj}, a_0^{sj} and a_n^{sj}.

The affine current $E^\alpha(z)$ can be expressed as follows:

$$E^\alpha(z) =: \mathrm{e}^{\frac{\mathrm{i}}{k}\tilde{\alpha}\cdot\tilde{\varphi}(z)} : r^\alpha\tilde{\mathcal{E}}^\alpha(z) \qquad (44)$$

where

$$r^\alpha = \mathrm{e}^{-\frac{\mathrm{i}}{k}\tilde{\alpha}x} \qquad x^A = \sum_{j=1}^k x^{Aj} \qquad (45)$$

and

$$\tilde{\varphi}^A = \sum_{j=1}^{k} \varphi^{Aj}. \tag{46}$$

These equations can be used as the generating equations for the $g/u(1)^d$ parafermions.

In the generating equations of [13] the factor $: \mathrm{e}^{\frac{\mathrm{i}}{k}\tilde{\alpha}\cdot\tilde{\varphi}(z)}:$ stands on the right. However it is not quite clear for us how to carry this factor to the right through c_α^j in equation (44).

In the case of $su(2)$ the cocycles c_α^j can be replaced by the identity operator. Since x^1 commutes with $\tilde{\mathcal{E}}^\alpha(z)$ we can set $x^1 = 0$ and rewrite equation (44) in the form [3]

$$E^+(z) = \sqrt{k}\psi^-(z) : \mathrm{e}^{\mathrm{i}\frac{\sqrt{2}}{k}\tilde{\varphi}(z)} : \qquad E^-(z) = \sqrt{k}\psi^+(z) : \mathrm{e}^{-\mathrm{i}\frac{\sqrt{2}}{k}\tilde{\varphi}(z)} : \tag{47}$$

where

$$\psi^-(z) = \frac{1}{\sqrt{k}}\tilde{\mathcal{E}}^+(z) \qquad \psi(z) = \frac{1}{\sqrt{k}}\tilde{\mathcal{E}}^-(z) \tag{48}$$

and $\tilde{\varphi}(z) \equiv \tilde{\varphi}^1(z)$.

These results can be generalized to non-simply-laced algebras using the vertex operator representation of the associated affine Lie algebras [14, 15].

6. Classical parafermions and the Dirac-bracket formalism

Quantization of the $g/u(1)^d$ theory can also be performed by using the Dirac-bracket formalism. The approach presented above is equivalent to the canonical quantization of the classical WZNW theory.

To show this we first find a classical system which corresponds to the $g/u(1)^d$ theory. Let us replace the operators $H^s(z)$, $E^\alpha(z)$ and $\phi^A(z)$ by the functions $H^s(\sigma)$, $E^\alpha(\sigma)$, $\phi^A(\sigma)$, satisfying the Poisson bracket relations

$$\begin{aligned} &\{H^s(\sigma), H^t(\sigma')\} = -k\delta^{st}\delta' \qquad \{H^s(\sigma), E^\alpha(\sigma')\} = \alpha^s E^\alpha(\sigma)\delta \\ &\{E^\alpha(\sigma), E^\beta(\sigma')\} = \begin{cases} f(\alpha,\beta)E^{\alpha+\beta}(\sigma)\delta & \text{if} \quad \alpha+\beta \text{ is a root} \\ -k\delta' + \alpha H(\sigma)\delta & \text{if} \quad \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{49}$$

$$\{\phi^A(\sigma), \phi^B(\sigma')\} = -\frac{k}{2}\delta^{AB}\epsilon \tag{50}$$

where $\delta = \delta(\sigma - \sigma')$ and $\epsilon = \epsilon(\sigma - \sigma')$. From this it follows that the constraints $H^A(\sigma) \approx 0$ are second class. The abelian operator constraints $\tilde{H}^A(z)$ are replaced by $\tilde{H}^A(\sigma) = H^A(\sigma) + \partial_\sigma\phi^A(\sigma)$, which satisfy the relations

$$\{\tilde{H}^A(\sigma), \tilde{H}^B(\sigma')\} = 0. \tag{51}$$

It it easy to check that the functions

$$\mathcal{H}^P(\sigma) = H^P(\sigma) \qquad \mathcal{E}^\alpha(\sigma) = E^\alpha(\sigma)\mathrm{e}^{\frac{1}{k}\tilde{\alpha}\cdot\phi(\sigma)} \tag{52}$$

commute with $\tilde{H}^A(\sigma)$:

$$\{\tilde{H}^A(\sigma), \mathcal{H}^P(\sigma')\} = 0 \qquad \{\tilde{H}^A(\sigma), \mathcal{E}^\alpha(\sigma')\} = 0. \tag{53}$$

Using the Poisson bracket relations (49) and (50) we find

$$\{\mathcal{H}^P(\sigma), \mathcal{H}^Q(\sigma')\} = -k\delta^{PQ}\delta' \qquad \{\mathcal{H}^P(\sigma), \mathcal{E}^\alpha(\sigma')\} = \alpha^P \mathcal{E}^\alpha(\sigma)\delta$$

$$\{\mathcal{E}^\alpha(\sigma), \mathcal{E}^\beta(\sigma')\} = \begin{cases} f(\alpha,\beta)\mathcal{E}^{\alpha+\beta}(\sigma)\delta & \\ -\dfrac{\tilde{\alpha}\tilde{\beta}}{2k}\mathcal{E}^\alpha(\sigma)\mathcal{E}^\beta(\sigma')\epsilon & \text{if} \quad \alpha+\beta \text{ is a root} \\ -k\delta' + \tilde{\alpha}\tilde{H}(\sigma)\delta + \tilde{\alpha}^\perp \cdot \mathcal{H}(\sigma)\delta & \\ +\dfrac{\tilde{\alpha}^2}{2k}\mathcal{E}^\alpha(\sigma)\mathcal{E}^{-\alpha}(\sigma')\epsilon & \text{if} \quad \alpha+\beta = 0 \\ -\dfrac{\tilde{\alpha}\tilde{\beta}}{2k}\mathcal{E}^\alpha(\sigma)\mathcal{E}^\beta(\sigma')\epsilon & \text{otherwise} \end{cases} \tag{54}$$

where $\mathcal{H}(\sigma) = (\mathcal{H}^P(\sigma))$. A similar algebra was found by the authors of [16].

The algebra Ω_{cl} defined by equations (51), (53) and (54) can be obtained using the Dirac-bracket formalism. Let $I(\sigma)$ and $J(\sigma)$ belong to the original current algebra (49). In the constrained theory they are replaced by the currents which satisfy the Dirac-bracket relations

$$\{I, J\}_D = \{I, J\} - \sum_{A,B} \int \mathrm{d}\sigma\, \mathrm{d}\sigma' \{I, H^A(\sigma)\}\{H^A(\sigma), H^B(\sigma')\}^{-1}\{H^B(\sigma'), J\}. \tag{55}$$

We find that H^s and E^α obey the algebra

$$\{H^A(\sigma), H^B(\sigma')\}_D = \{H^A(\sigma), H^P(\sigma')\}_D = \{H^A(\sigma), E^\alpha(\sigma')\}_D = 0$$

$$\{H^P(\sigma), H^Q(\sigma')\}_D = -k\delta^{PQ}\delta' \qquad \{H^P(\sigma), E^\alpha(\sigma')\}_D = \alpha^P E^\alpha(\sigma)\delta$$

$$\{E^\alpha(\sigma), E^\beta(\sigma')\}_D = \begin{cases} f(\alpha,\beta)E^{\alpha+\beta}(\sigma)\delta & \\ -\dfrac{\tilde{\alpha}\tilde{\beta}}{2k}E^\alpha(\sigma)E^\beta(\sigma')\epsilon & \text{if} \quad \alpha+\beta \text{ is a root} \\ -k\delta' + \alpha H(\sigma)\delta & \\ +\dfrac{\tilde{\alpha}^2}{2k}E^\alpha(\sigma)E^{-\alpha}(\sigma')\epsilon & \text{if} \quad \alpha+\beta = 0 \\ -\dfrac{\tilde{\alpha}\tilde{\beta}}{2k}E^\alpha(\sigma)E^\beta(\sigma')\epsilon & \text{otherwise.} \end{cases} \tag{56}$$

This algebra is isomorphic to Ω_{cl}. The corresponding map is

$$H^A(\sigma) \to \tilde{H}^A(\sigma) \qquad H^P(\sigma) \to \mathcal{H}^P(\sigma) \qquad E^\alpha(\sigma) \to \mathcal{E}^\alpha(\sigma). \tag{57}$$

To quantize the system we could therefore start from the Dirac brackets (56) (or equivalently from the Poisson brackets (51), (53) and (54)), use ansatz (52) and replace the functions $H^s(\sigma)$, $E^\alpha(\sigma)$ and $\phi^A(\sigma)$ by the corresponding operators.

7. Conclusion

Treating the $g/u(1)^d$ conformal field theory as the WZNW theory for $\hat{g}$ with the constrained $\hat{u}(1)^d$ currents and using a modification of the generalized canonical quantization method we have found the explicit construction of the symmetry generators of the $g/u(1)^d$ theory. The presented quantization may be generalized to the WZNW theory for $\hat{g}$ with an arbitrary constrained current algebra $\hat{h} \subset \hat{g}$. We expect that in the general case the conformal invariant solution of the generating equations determines a symmetry algebra of the g/h conformal field theory.

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